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A Novel Application of the Abaoub-Shkheam Decomposition Method to Nonlinear Fractional Diffusion-Wave Equations

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Abstract: In this paper, a nonlinear fractional diffusion and wave equations are solved using the Abaoub-Shkheam Decomposition Method (QDM). The Caputo sense is used to characterise the fractional derivative. An example is provided to demonstrate the method's efficiency and practicality.

Keywords: Abaoub Shkheam transform, Adomian decomposition method, Fractional derivatives, A nonlinear fractional diffusion and wave equations.

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I. INTRODUCTION

Fractional differential equations have acquired important regard in the latest years due to their capability to model complex phenomena in several scientific and engineering fields, including fluid dynamics, viscoelasticity, signal processing, and singular diffusion. In particular, fractional diffusion-wave equations have been widely studied as they provide a more precise description of wave propagation and diffusion processes in heterogeneous and memory-dependent media. Yet, the nonlinear nature of these equations presents a great challenge in finding exact or approximate solutions.

Many analytical and numerical methods have been progressing to solve nonlinear fractional differential equations [6], including the Adomian Decomposition Method (ADM) [1], which gives a solution in terms of a quickly convergent power series, the Homotopy Analysis Method (HAM), and the Laplace Transform Method. While these techniques have proven to be efficient in definite cases, they often face restrictions in handling complex nonlinearities and guaranteeing rapid convergence.

In the recent years, several authors, for example, Mainardi [3,4], Schneider and Wyss [5], and El-Sayed [2], have investigated the fractional diffusion-wave equation and its special properties. Fractional diffusion and wave equations have important applications to mathematical physics. Thus, the development of novel and efficient solution mechanisms remains a definitive area of research.

In this paper, we present and explore the Abaoub-Shkheam Decomposition Method (QDM) as a novel method for solving nonlinear fractional diffusion-wave equations. The QDM extends the ability of existing decomposition methods by coupling elements of integral transforms and series expansion techniques, allowing for a more efficient treatment of fractional-order operators and nonlinear terms. By applying this method, we aim to provide a methodical and credible approach for getting approximate analytical solutions to nonlinear fractional diffusion-wave models.

II. PRELIMINARIES

II.2. Basic Definitions

Definition II.1. [8]

A real function $f(t), t > 0$ is said to be in the space $C_\alpha, \alpha \in \Re$;if there exists areal number $(P > \alpha)$, such that $f(t) = t^P f_1(t)$.Where $f_1 \in C[0,\infty)$.
Clearly $C_\alpha \subset C_\beta$ if $\beta \leq \alpha$.

Definition II.2. [8]

A function $f(t), t > 0$ is said to be in the space C_α^m ; $m \in N \cup \{0\}$ if $f^{(m)} \in C_\alpha$.

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Definition II.3. [9]

The Riemann-Liouville (R-L) fractional integral of order $\alpha > 0$ of a function $f \in C_\alpha$ is defined as

$$ {}_aI_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(\tau)}{(t-\tau)^{-\alpha+1}} d\tau, \quad t > 0. $$

Definition II.4. [9]

The Caputo fractional derivatives of f(t)order $\alpha > 0$ is defined as

$$ {}_a^CD_t^\alpha f(t) = {}_aI_t^{n-\alpha} f^{(n)}(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, & n-1 < \alpha < n \\ f^{(n)}(t) & , \alpha = n, \end{cases} $$

where $n \in \mathbb{N}$.

The following are the basic properties of the operator ${}_a^CD_t^\alpha$:

1. ${}_a^CD_t^\alpha \, {}_aI_t^\alpha f(t) = f(t)$.
2. ${}_aI_t^\alpha \, {}_a^CD_t^\alpha f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0) \frac{t^k}{k!}$.

Definition II.5. [7]

The Abaoub Shkheam transform is defined over the set of function

$$ \mathcal{B} = \left\{ f(t) : \exists\, N, k_1, k_2 > 0 \, , |f(t)| < Ne^{\left(\frac{|t|}{k_j}\right)}, j = 1,2;\ t \in (-1)^j \times [0, \infty), \right\} $$

by the following formula

$$ Q\{f(t)\} = \lim_{b \to \infty} \int_0^b f(ut) e^{\frac{-t}{s}} dt = T(u, s). $$

II.2. Abaoub Shkheam transform of Caputo Fractional Derivative

Theorem. 1.

The Abaoub Shkheam transform of the Caputo fractional derivative is defined as

$$ Q\left\{ {}_a^CD_t^\alpha (f(t)) \right\} = \frac{T(v,s)}{v^\alpha s^\alpha} - \frac{1}{v} \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{(vs)^{\alpha-k-1}} $$

where $n-1 < \alpha < n$, $n \epsilon \mathbb{Z}^+$.

Proof

Since

$$ {}_a^CD_t^\alpha (f(t)) = {}_aI_t^{n-\alpha} f^{(n)}(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)\, d\tau, $$

Setting $g(t) = f^{(n)}(t)$, and applying Abaoub Shkheam transform on both sides of above equation

$$ Q\left\{ {}_a^CD_t^\alpha (f(t)) \right\} = Q \left\{ \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} g(\tau)\, d\tau \right\} = $$

$$ = \frac{1}{\Gamma(n-\alpha)} v\, Q\{t^{\,n-\alpha-1}\}\, Q\{g(t)\} = v^{n-\alpha} s^{n-\alpha} Q\{f^{(n)}(t)\} $$

$$ = \frac{T(v,s)}{v^\alpha s^\alpha} - \frac{1}{v} \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{(vs)^{\alpha-k-1}}. $$

III. ANALYSIS OF THE METHOD

Consider the following general form of nonlinear fractional partial differential equation :

$$ D_t^\alpha u(\boldsymbol{x}, t) = \sum_{i=1}^{n} N_i(\boldsymbol{x}, t) \frac{\partial^2 u(\boldsymbol{x}, t)}{\partial x_i^2} + \emptyset(\boldsymbol{x}, t) u^m(\boldsymbol{x}, t), \qquad (III.1) $$

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where $m = 2,3, \dots,$ $\boldsymbol{x} = (x_1, x_2, \dots, x_n)$; D_t^α denotes the Caputo fractional derivative and $N_i(\boldsymbol{x}, t) \in C_\alpha$. For $m = 0$ or 1 and $0 < \alpha < 1$, $(III.1)$ represents a nonlinear fractional diffusion equation, and $1 < \alpha < 2$, $(III.1)$ represents a nonlinear fractional wave equation (homogeneous if $\emptyset(\boldsymbol{x}, t) = 0$ and nonhomogeneous otherwise).

III.1. The Abaoub Shkheam decomposition Method For SolvingA nonlinear fractional diffusion-wave equation

Consider the following general form of a nonlinear fractional diffusion-wave equation with the specified initial condition:

$$D_t^\alpha u(\boldsymbol{x}, t) = \sum_{i=1}^{n} N_i(\boldsymbol{x}, t) \frac{\partial^2 u(\boldsymbol{x}, t)}{\partial x_i^2} + \emptyset(\boldsymbol{x}, t) u^m(\boldsymbol{x}, t), \qquad (III.2)$$

$$u(\boldsymbol{x}, 0) = f(\boldsymbol{x}), \qquad 0 < \alpha < 1 \qquad (III.3)$$

$$u(\boldsymbol{x}, 0) = f(\boldsymbol{x}), \quad \frac{\partial u(\boldsymbol{x}, 0)}{\partial t} = g(\boldsymbol{x}), \quad 1 < \alpha < 2 \qquad (III.4)$$

The process starts by applying the Q-transform to both sides of Equation $(III.2)$.

$$Q\{D_t^\alpha u(\boldsymbol{x}, t)\} = Q\left\{\sum_{i=1}^{n} N_i(\boldsymbol{x}, t) \frac{\partial^2 u(\boldsymbol{x}, t)}{\partial x_i^2}\right\} + Q\{\emptyset(\boldsymbol{x}, t) u^m(\boldsymbol{x}, t)\}. \qquad (III.5)$$

Case (i) $0 < \alpha < 1$, then by theorem (1), and $(III.5)$, we get

$$Q\{u(\boldsymbol{x}, t)\} = \frac{s^\alpha v^\alpha}{v}\left\{\frac{f(\boldsymbol{x})}{(s\, v)^{\alpha-1}}\right\} + v^\alpha s^\alpha Q\left\{\sum_{i=1}^{n} N_i(\boldsymbol{x}, t) \frac{\partial^2 u(\boldsymbol{x}, t)}{\partial x_i^2}\right\} + v^\alpha s^\alpha Q\{\emptyset(\boldsymbol{x}, t) u^m(\boldsymbol{x}, t)\}. \qquad (III.6)$$

Using the inverse of Abaoub Shkheam transform on both sides of Eq. $(III.6)$ gives

$$u(\boldsymbol{x}, t) = Q^{-1}\{s\, f(\boldsymbol{x})\} + Q^{-1}\left\{v^\alpha s^\alpha Q\left[\sum_{i=1}^{n} N_i(\boldsymbol{x}, t) \frac{\partial^2 u(\boldsymbol{x}, t)}{\partial x_i^2}\right]\right\} + + Q^{-1}[v^\alpha s^\alpha Q\ \{\emptyset(\boldsymbol{x}, t) u^m(\boldsymbol{x}, t)\}]\,. \qquad (III.7)$$

The next step is using the Adomian decomposition method that represent the solution $u(\boldsymbol{x}, t)$as an infinite series given by

$$u(\boldsymbol{x}, t) = \sum_{n=0}^{\infty} u_n\,(\boldsymbol{x}, t), \qquad (III.8)$$

and the nonlinear term is decomposed as follows:

$$\emptyset(\boldsymbol{x}, t) u^m(\boldsymbol{x}, t) = \sum_{k=0}^{\infty} A_k, \qquad (III.9)$$

where $A_k, k \geq 0$ are the Adomian polynomials given by:

$$A_k = \frac{1}{k!} \frac{d^k}{d\gamma^k}\left[\left(\sum_{n=0}^{k} u_n(\boldsymbol{x}, t) \gamma^n\right)^m\right]_{\gamma=0}, k \geq 0$$

Substituting in $(III.7)$, we get

$$\sum_{n=0}^{\infty} u_n\,(\boldsymbol{x}, t) = Q^{-1}\{s\, f(\boldsymbol{x})\} + Q^{-1}\left\{v^\alpha s^\alpha Q\left[\sum_{i=1}^{n} N_i(\boldsymbol{x}, t) \frac{\partial^2}{\partial x_i^2} \sum_{n=0}^{\infty} u_n\,(\boldsymbol{x}, t)\right]\right\} + + Q^{-1}\left[v^\alpha s^\alpha Q\left\{\sum_{k=0}^{\infty} A_k(u_0, u_1, \dots, u_k)\right\}\right],$$

where the components $u_n(\boldsymbol{x}, t), n \geq 0$ will be determined in a recursive manner

$$u_0(\boldsymbol{x}, t) = Q^{-1}\{s f(\boldsymbol{x})\}$$

$$u_{n+1}(\boldsymbol{x}, t) = Q^{-1}\left\{v^\alpha s^\alpha Q\left[\sum_{i=1}^{n} N_i(\boldsymbol{x}, t) \frac{\partial^2}{\partial x_i^2} \sum_{n=0}^{\infty} u_n\,(\boldsymbol{x}, t)\right]\right\} + Q^{-1}\left[v^\alpha s^\alpha Q\left\{\sum_{k=0}^{\infty} A_k(u_0, u_1, \dots, u_k)\right\}\right]$$

Case (ii) if $1 < \alpha < 2$, then by theorem (1), and $(III.5)$, we get

$$Q\{u(\boldsymbol{x}, t)\} = s\, u(\boldsymbol{x}, 0) + v s^2 \frac{\partial u(\boldsymbol{x}, 0)}{\partial t} + v^\alpha s^\alpha Q\left\{\sum_{i=1}^{n} N_i(\boldsymbol{x}, t) \frac{\partial^2 u(\boldsymbol{x}, t)}{\partial x_i^2}\right\} + v^\alpha s^\alpha Q\{\emptyset(\boldsymbol{x}, t) u^m(\boldsymbol{x}, t)\}, \qquad (III.10)$$

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Using the inverse of Abaoub Shkheam transform on both sides of Eq. $(III.10)$ gives

$$u(\boldsymbol{x},t) = Q^{-1}\{s\,f(\boldsymbol{x})\} + Q^{-1}\{vs^2 g(\boldsymbol{x})\} + Q^{-1}\left\{v^\alpha s^\alpha Q\left[\sum_{i=1}^{n} N_i(\boldsymbol{x},t)\frac{\partial^2 u(\boldsymbol{x},t)}{\partial x_i^2}\right]\right\} + + Q^{-1}[v^\alpha s^\alpha Q\,\{\emptyset(\boldsymbol{x},t)u^m(\boldsymbol{x},t)\}]. \quad (III.11)$$

Substituting in $(III.11)$, we get

$$\sum_{n=0}^{\infty} u_n(\boldsymbol{x},t) = Q^{-1}\{s\,f(\boldsymbol{x})\} + Q^{-1}\{vs^2 g(\boldsymbol{x})\} + Q^{-1}\left\{v^\alpha s^\alpha Q\left[\sum_{i=1}^{n} N_i(\boldsymbol{x},t)\frac{\partial^2}{\partial x_i^2}\sum_{n=0}^{\infty} u_n(\boldsymbol{x},t)\right]\right\} + + Q^{-1}\left[v^\alpha s^\alpha Q\left\{\sum_{k=0}^{\infty} A_k(u_0,u_1,\dots,u_k)\right\}\right],$$

Following the Abaoub Shkheam decomposition method, define

$$u_0(\boldsymbol{x},t) = Q^{-1}\{sf(\boldsymbol{x})\} + Q^{-1}\{vs^2 g(\boldsymbol{x})\}$$

$$u_{n+1}(\boldsymbol{x},t) = Q^{-1}\left\{v^\alpha s^\alpha Q\left[\sum_{i=1}^{n} N_i(\boldsymbol{x},t)\frac{\partial^2}{\partial x_i^2}\sum_{n=0}^{\infty} u_n(\boldsymbol{x},t)\right]\right\} + Q^{-1}\left[v^\alpha s^\alpha Q\left\{\sum_{k=0}^{\infty} A_k(u_0,u_1,\dots,u_k)\right\}\right].$$

III.2. Illustrative Examples

We introduce some examples to provide a comprehensive overview of this method

Example. 1.

Consider the fractional wave equation

$$D_t^\alpha u - u_{xx} - u + u^2 = xt + x^2t^2, \quad 0 < x < 1, \quad t > 0 \quad (III.12)$$

$$u(\boldsymbol{x},0) = 1, \quad \frac{\partial u(\boldsymbol{x},0)}{\partial t} = x, \quad 1 < \alpha < 2. \quad (III.13)$$

Taking Abaoub – Shkheam transform of $(III.12)$ we have

$$Q\{D_t^\alpha u\} = Q\{u_{xx} + u - u^2 + xt + x^2t^2\},$$

by theorem (1), and using conditions $(III.13)$, we get

$$Q\{u(\boldsymbol{x},t)\} = s.1 + vs^2.x + v^\alpha s^\alpha Q\{u_{xx} + u - u^2 + xt + x^2t^2\}$$

$$Q\{u(\boldsymbol{x},t)\} = s + vs^2.x + v^\alpha s^\alpha Q\{u_{xx} + u - u^2 + xt + x^2t^2\}.$$

Using the inverse of Abaoub Shkheam transform on both sides of above equation gives

$$u(\boldsymbol{x},t) = Q^{-1}\{s + vs^2.x + v^\alpha s^\alpha Q\{u_{xx} + u - u^2 + xt + x^2t^2\}\}. \quad (III.14)$$

Substituting by $(III.8)$ and $(III.9)$ in $(III.14)$, we get

$$\sum_{n=0}^{\infty} u_n(\boldsymbol{x},t) = Q^{-1}\left\{s + vs^2.x + v^\alpha s^\alpha Q\left\{\frac{\partial^2}{\partial x^2}\sum_{n=0}^{\infty} u_n(\boldsymbol{x},t) + \sum_{n=0}^{\infty} u_n(\boldsymbol{x},t) - \sum_{k=0}^{\infty} A_k(u_0,u_1,\dots,u_k) + xt + x^2t^2\right\}\right\}$$

Following the Abaoub Shkheam decomposition method, define

$$u_0(\boldsymbol{x},t) = Q^{-1}\{s\} + xQ^{-1}\{vs^2\},$$

$$u_{k+1}(\boldsymbol{x},t) = Q^{-1}\left\{(v^\alpha s^\alpha)Q\left\{\frac{\partial^2 u_k(\boldsymbol{x},t)}{\partial x^2} + u_k - A_K + xt + x^2t^2\right\}\right\}.$$

Consequently

$$u_0(\boldsymbol{x},t) = 1 + xt.$$

$$u_1(\boldsymbol{x},t) = Q^{-1}\left\{(v^\alpha\, s^\alpha)Q\left\{\frac{\partial^2 u_0(\bar{x},t)}{\partial x^2} + u_0 - A_0 + xt + x^2t^2\right\}\right\}$$

$$u_1(\boldsymbol{x},t) = Q^{-1}\left\{(v^\alpha\, s^\alpha)Q\left\{\frac{\partial^2}{\partial x^2}(1+xt) + (1+xt) - (1+xt)^2 + xt + x^2t^2\right\}\right\} = 0$$

$$u_{k+1}(\boldsymbol{x},t) = 0, \quad For\ all\ \ k \geq 0.$$

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Hence

$$u(\boldsymbol{x},t) = \sum_{n=0}^{\infty} u_n\,(\boldsymbol{x},t) = 1 + xt.$$

IV. CONCLUSION

Fractional diffusion equations can be solved effectively and efficiently with the Abaoub–Shkheam Decomposition Method (QDM). It offers quickly convergent series solutions without the computational difficulties of numerical approaches by fusing the Abaoub–Shkheam transform with the Adomian decomposition method.

The approach is a useful tool in mathematical physics, as demonstrated by the examples, which demonstrate its accuracy and versatility for nonlinear problems. This method's potential for wider applications in science and engineering is demonstrated by its ability to be adapted to more complicated systems.

REFERENCES

[1]. G. Adomian, Solving Frontier Problems of Physics: The Decomposition Method, Kluwer, 1994..
[2]. A. El-Sayed , “Fractional-order diffusion-wave equation” Int. J. Theor. Phys. Vol. 35; Issue 2; pp. 311-322.1996.
[3]. F. Mainardi, On the initial value problem for the fractional diffusion-wave equation, in: S. Rionero, T. Ruggeeri (Eds.), Waves and Stability in Continuous Media, World Scientific, Singapore, pp. 246–251. 1994.
[4]. F. Mainardi, Fundamental solutions for the fractional diffusion-wave equation, Appl. Math. Lett. 9 (1996) 23–28.
[5]. W.R. Schneider, W. Wyss, Fractional Diffusion and Wave Equations. J. Math. Phys. 30 (1) 134–144.1998. https://doi.org/10.1063/1.528578
[6]. K. Diethelm, An algorithm for the numerical solution of differential equations of fractional order, Electron. Trans. Numer. Anal. 5 (1997) 1–6.
[7]. A. Abaoub , A. Shkheam , “ The new Integral transform : Abaoub Shkheam transform” Iaetsd journal for advanced research in applied sciences, ISSN NO: 2394-8442; vol. VII, Issue VI, pp. 8 – 14.2020.
[8]. H. Jafari, V. Daftardar-Gejji “Solving linear and nonlinear fractional diffusion and wave equations by Adomian decomposition” Applied Mathematics and Computation; Vol. 180 ; pp. 488–497.2006.
[9]. A. Kilbas and J. Trujillo “On Caputo and Riemann-Liouville derivatives of fractional order” Computers and Mathematics with Applications, Vol. 62;, pp. 1602–1611.2011.