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On Left Permitivity over a Matrix Ring's and Module's

Asha Saraswathi. B[1], Upase Rajashekhar[2]

Professor and H.O.D, Dep't of Mathematics, Srinivas University, Srinivas Nagar, Mukka. Surathkal, MAngaluru-574146.[1]

Faculty Member's, Upase Education Institute, Jayanagar, Dharwad-580001, Karnataka[2]

Abstract: Aim of this paper a ring R be an associative ring with identity and all modules are unitary R_n and $J(R)$ are denotes the matrix rings and Jacobson radical and the singular left ideal of R, also $\aleph$ be maximal left ideal of R, if R is left primitive, $e(\neq 0)$ is idempotent in R then eRe is left primitive.

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Keywords: R matrix ring,, R_n finite matrix ring,. $\aleph$ be maximal left ideal, $J(R)$ Jacobson radical and eRe is left primitive.

INTRODUCTION

In the year 1951, E.C. Posner .al.[1], defined let R be an associative ring with identity and all modules are unitary, R_n *and* $J(R)$ *are denotes the matrix rings and Jacobson radical and the singular left ideal of R, author* was first to prove this, his proof was very difficult and complicated. His proof is purely ring-theoretic without bringing in modules, whether his Corollary 5.12 is true without assuming the ring R has identity element. Actually, the answer is "yes".

Definition: 1.1. For a ring R is said to be left-primitive , the following conditions holds;

(i) if there exists a maximal left ideal I of R such that $(I:R)=(0)$
(ii) if there exists a simple faith full $R-$ module.

Definition: 1.2. Let R be a ring. A non empty subset I of R is said to be a **'lift ideal'** of R. If the following conditions holds

(i) I is a subgroup of the additive group of R, that is $\forall a,b\in I \Rightarrow a-b\in I$
(ii) $\forall a\in I,\ r\in R \Rightarrow ra\in I$

Definition: 1.3. A left ideal I of a ring R is said to be a **"maximal left ideal of a ring** R **"**, if the following conditions holds;

(i) $I\neq R$
(ii) There are no left ideal of R between I and R. That is if J a left ideal of R, such that $I\subseteq J\subseteq R$ then either $J=I$ or $J=R$.

Example: 1.4. Every field, and more generally, every division ring is semi primitive. For, if D is any division ring, it has no nontrivial ideals. Suppose $J(D)=D$ .then $1\in J(D)$ which implies, by definition, $1-1=0$, right invertible in R which is impossible. So, $J(D)=0$. Thus, D is semi primitive.

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Example: 1.5. The ring ℤ of integers is semi primitive. For, let $a \in J(\mathbb{Z})$. Then $1-a$ is a unit in ℤ. Since 1 and –1 are the only units in ℤ, it follows that either $1-a=1$ or -1. If $1-a=1$, then $a=0$. Suppose $1-a=-1$. Then $a=2$. So 2 $\in J(\mathbb{Z})$. Hence $1-4=-3$ is a unit in ℤ which is absurd. Hence $J(\mathbb{Z})$ = 0. So ℤ is semi primitive.

We have the following proposition as an easy consequence of Proposition.

Proposition: 1.6. A ring R is Simi primitive if and only if the matrix ring $M_n(R)$ (for any positive integer n) is semi primitive

Proof: Obvious

Proposition: 1.7. If R is a ring with an identity and let $n \in N$ be an integer then proves that $E_{11}M_n(R)E_{11} \cong R$.

Proof: We first compute the elements of the sub − ring $E_{11}M_n(R)E_{11}$ of $M_n(R)$. Let $A=(a_{ij}) \in M_n(R)$, then for $1 \le i,\ j \le n$.

Let $$E_{11}AE_{11} = E_{11}((a_{ij}))E_{11} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} 1 & \cdots & 0 \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} = a_{11}E_{11}$$

Case (i). If, $(i,j)=(1,1)$, $(E_{11}AE_{11})_{ij} = (E_{11}AE_{11})_{11} = \sum_{k=1}^{n}\sum_{l=1}^{n}(E_{11})_{1k}\,a_{kl}\,(E_{11})_{l1} = \sum_{l=1}^{n}(E_{11})_{1l}\,a_{ll}\,(E_{11})_{l1} = 1(a_{11})1 = a_{11}$

Case (ii), if $(i,j) \neq (1,1)$, In this case either $i \neq 1$, or $j \neq 1$, we assume that $i \neq 1$,

$$(E_{11}AE_{11})_{ij} = \sum_{k=1}^{n}\sum_{l=1}^{n}(E_{11})_{ik}\,a_{kl}\,(E_{11})_{lj} = \sum_{l=1}^{n}(E_{11})_{il}\,a_{ll}\,(E_{11})_{lj} = 0(a_{11})1 = 0$$

$$\therefore \quad E_{11}AE_{11} = \begin{cases} a_{11} & if \quad (i,j)=(1,1) \\ 0 & if \quad (i,j)\neq(1,1) \end{cases}$$

Now we define a map, $\phi: E_{11}M_n(R)E_{11} \to R$ as follows, Let, $B \in E_{11}M_n(R)E_{11}$ then $B = E_{11}AE_{11}$, for some $A \in M_n(R)$ then $B = a_{11}E_{11}$ were $A=(aij)$, we define, $\phi(B)=a_{11}$. We prove ϕ is well defined: Let, $A=(a_{ij})$ and $C=(c_{ij})\} \in M_n(R)$, $\forall\ 1 \le i, j \le n$. Since, $E_{11}AE_{11} = E_{11}CE_{11} \Rightarrow a_{11}E_{11} = c_{11}E_{11} \Rightarrow a_{11} = c_{11}$. Therefore $\phi(A)=\phi(C)$. Hence, ϕ is well defined and it is easy to check that Φ is ring homomorphism. We show that ϕ is additive:

Since, let $E_{11}AE_{11} + E_{11}CE_{11} = E_{11}(A+C)E_{11}$ implies $\phi(E_{11}AE_{11} + E_{11}CE_{11}) = \phi(E_{11}(A+C)E_{11})$ $= (a_{11}+c_{11}) = \iota(E_{11}AE_{11}) + \phi(E_{11}CE_{11})$ and also we claim ϕ is multiplicative: Let $A, C \in M_n(R)$, Let $E_{11}AE_{11} \cdot E_{11}CE_{11} = E_{11}(AE_{11}C)E_{11}$ this implies
$\phi(E_{11}AE_{11} \cdot E_{11}CE_{11}) = \phi[E_{11}(AE_{11}C)E_{11}] = a_{11}c_{11} = \phi(E_{11}AE_{11})\phi(E_{11}CE_{11})$, **$\phi$: is one − one:** Enough to prove that, $Ker(\phi)=\{0\}$. Let $A \in M_n(R)$ such that $\phi(E_{11}AE_{11})=0$ $\Rightarrow E_{11}AE_{11}=0 \Rightarrow a_{11}E_{11}=0 \Rightarrow a_{11}=0$. Hence, $Ker(\phi)=\{0\}$. Hence, ϕ is one − one.

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ϕ **: is onto,** Let, $a \in R$, Define $a_{ij} = \begin{cases} a \ \ if \ \ (i,j)=(1,1) \\ 0 \ \ if \ \ (i,j)\neq(1,1) \end{cases}$ then $A=(a_{ij})\in M_n(R)$

Let, $B = aE_{11}$ then $B = E_{11}AE_{11}$, where $A = aE_{11} \in M_n(R)$, let $E_{11}(aE_{11})E_{11} = aE_{11} = B$ in $E_{11}M_n(R)E_{11}$, and $\phi(B) = a_{11} = a$. Thus, ϕ is a ring is isomorphism from the ring $E_{11}M_n(R)E_{11}$ onto the ring R. So, $E_{11}M_n(R)E_{11} \cong R$.

Theorem: 1. 8. Suppose R is primitive ring then there is a division ring D such that, either

i) R is isomorphic with the ring $D_n = M_n(D)$ of all $n\times n$ matrices entries from D, for some n. Or

ii) There is an ascending chain $R_1 \subseteq R_2 \subseteq \ldots$ of sub ring of R and for each k an isomorphism $f_k : R_k \to D_k = M_k(D)$.

Proof: obvious

Theorem: 1.9. If 'e' is a nonzero idempotent in a left primitive ring R then $e\mathrm{Re}$ is also a left primitive ring.

Proof: Since, by hypothesis, R is right primitive, there exists a simple, faithful $R-$module, say, M. We now prove that eM is a simple, faithful $e\mathrm{Re}$ module.

(i) Since $(e\mathrm{Re})eM = e\mathrm{R}(eeM) = e\mathrm{Re}M \subseteq M$, eM is a left $e\mathrm{Re}-$ module, Also since M is faithful as an $R-$ module, $eM \neq 0$.

(ii) eM is simple : By Proposition 2.19, we need only prove that $(e\mathrm{Re})x = eM$ for each nonzero x in eM. So, let $x \neq 0 \in eM(\subseteq M)$.Then $x = ex$ .Since M is simple by Proposition we have $Rx = M$. Hence $(e\mathrm{Re})x = eRx = eM$. Thus, eM is simple as an $e\mathrm{Re}-$ module.

(iii) eM is faithful: Let $a \in e\mathrm{Re}$ be such that $(eM)a = 0$. Since M is faithful as an $R-$module, it follows that $ae = a = 0$. Thus eM is faithful. So, eM is a simple, faithful $e\mathrm{Re}-$ module and hence $e\mathrm{Re}$ is a left primitive ring.

Before stating the corollary1.11, we note that permittivity is preserved under isomorphism. For, if $f : R \to R^1$ is an isomorphism from a ring R onto a ring R^1 and if M is a simple, faithful $R-$ module, then M with the induced structure of R^1- module given by $r^1m = rm$ for $m \in M, r^1 \in R^1$, (note that for each $r^1 \in R^1$ there is unique $r \in R$ such that $f(r) = r^1$) is also a simple, faithful R^1- module .

Corollary: 1.10. If R is a ring with identity and 'n' is a positive integer such that the matrix ring $M_n(R)$ is left primitive then R itself is left primitive.

Proof: Follows from Theorem 1.12 and Theorem1.13 and the fact that permittivity is preserved under isomorphism's.

We now prove the converse of the above corollary in the general case, that is, without assuming the existence of the identity element of the ring.

Theorem: 1.11. Let R be a ring and let $R_n = S$ be the matrix ring over R, then R is left primitive if and only if R_n is left primitive.

Proof: Only if: Since by hypothesis R is left primitive so then there exists a simple faithful left $R-$module say M. We construct a simple faithful left $S-$ module. Let M^n be the n^{th} direct power of M .We make M^n as a left $S-$ module, as follows: Let $M^n = \{(x_1, x_2, \cdots, x_n)/ x_i \in M, 1 \leq i \leq n\}, A = ((a_{ij})) \in S$.

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Define $AX=\left(\sum_{i=1}^{n}a_{i1}x_i,\cdots,\sum_{i=1}^{n}a_{in}x_i\right)\in M^n$ it is easy to check that M^n because a left $S-$module. ***To prove:*** M_S^n is faithful.

We need only prove that $ann_s(M^n)=(0)$, let $A\in ann_s(M^n)\Rightarrow AM^n=0$. Let $A=0$.For this we need to prove that $a_{ij}=0,\ \forall\ i,j=1,2,\ldots,n$. Let $1\le p,q\le n$.We prove that $a_{pq}=0$. Since by hypothesis M_R is faithful. **It suffices to prove that** $Ma_{pq}=0$**:**

Let $m\in M$ and let $x=(x_1,x_2,\cdots,x_n)\in M^n$ where $x_i=0,\ \forall\ i\ne p\ and\ x_p=m$. By hypothesis $AX=0$, but $AX=\left(\sum_{i=1}^{n}a_{i1}x_i,\cdots,\sum_{i=1}^{n}a_{iq}x_i,\cdots,\sum_{i=1}^{n}a_{in}x_i\right)$ implies $\sum_{i=1}^{n}a_{ij}x_i=0$ for $j=1,2,\ldots n$. In particular $\sum_{i=1}^{n}a_{pq}x_i=0$ $\Rightarrow a_{pq}\sum_{i=1}^{n}x_i=0\Rightarrow a_{pq}m=0$. Since, $m\in M$ was arbitrarily chosen, it follows that $ma_{pq}=0,\ \forall\ m\in M$ $\Rightarrow Ma_{pq}=0\Rightarrow a_{pq}=0$. Since M_R is faithful and since, $p,q\in\{1,2,\ldots,n\}$ where arbitrarily chosen it follows that $a_{ij}=0,\ \forall\ i,j=1,2,\ldots,n$ .Hence, $A=0$ so, M_S^n is faithful.. We Claim M_S^n is simple;

We first prove that since $M\ne(0)$. So M^n now we need only prove that $Sx=M^n\ \ \forall x\ne 0\in M^n$. So $\exists\ p\in\{1,2,\ldots,n\}$ such that $x_p\ne 0\in M$. Since M_R is simple, we get that $Rx_p=M$. We Claim $SX=M^n$. It's clearly $SX\subseteq M^n$. Let $Y=(y_1,y_2,\ldots,y_n)\in M^n$ .Now, for $1\le i\le n$, $y_i\in M=Rx_p$, So $y_i=r_ix_p$ for some $r_i\in R$, for $1\le i\le n$ .Define, $A=((a_{ij}))\in S$ as $a_{ij}=\begin{cases}0, & if\quad i\ne p\\ r_j, & if\quad i\ne p\qquad 1\le j\le n\end{cases}$ Then,

$$AX=\left(\sum_{i=1}^{n}a_{i1}x_i,\cdots,\sum_{i=1}^{n}a_{in}x_i\right)=\left(a_{p1}x_p,\cdots,a_{pn}x_p\right)=\left(r_1x_p,\cdots,rx_{pn}\right)=\left(y_1,y_2,\ldots,y_n\right)=Y.$$

. Thus, $Y=AX\in SX$, hence $M^n\subseteq S$. So, $SX=M^n$. Hence, M_S^n **is simple.** So, M^n is a simple faithful left $S-$module, so, $S=R_n$ is left primitive.

If: Claim: R has identity element in this case $E_{11}SE_{11}\,\square\,R$ because $E_{11}M_n(R)E_{11}=\left\{A\in M_n(R)\ /\ a_{ij}=0,\ \forall\ (i,j)\ne(1,1)\right\}$ .Now our assertion is trivial. **General Case:** Let M be a simple faithful left R_n-module, then

$i)\ R_nM=M\quad (\because by\ definition\ is\ simple\ {}_RM=M)$

ii) If $m\in M\quad and\quad R_nm=0,\quad then\quad m=0$

Let $1\le i\le n$ define $I_p=\left\{A=(a_{ij})\in R_n\ /\ a_{ij}=0,\ \forall\ i\ne p\right\}$. Clearly, I_p is a left ideal of R_n and $R_n=I_1+I_2+I_3+\ldots+I_n$. [$\because R$ is left primitive; $e=e^2\ne 0\in R$]$\Rightarrow eRe$ is left primitive; if ${}_RM$ is simple faithful. $eRe(eM)=(e\mathrm{Re}e)M\ =e(\mathrm{Re}M)\ \subseteq eM\quad(\because eM\ne 0)$

$a(eM)=0\Rightarrow aeM=aM\ \ (\because a\in eRe)$. Next define for $1\le q\le n$, $J_q=\left\{A=(a_{ij})\in R_n\ /\ a_{ij}=0\ \ \forall\ j\ne q\right\}$. Clearly; J_q is a left ideal of R_n and $R_n=J_1+J_2+J_3+\ldots+J_n$. Also $I_sJ_r=0$, $\forall\ r\ne s\in\{1,2,\ldots,n\}$ .The map $\phi:R\to R_n$ defined by, $\phi(r)=Ir$, $\forall r\in R$, is embedding of R into R_n, and then $R^*=\phi(R)$ is a sub ring of R_n isomorphic to R. So, we can identity R as a sub ring of R, with the scalar matrix Ir in R_n. Let $M_i=J_iM\quad for\quad 1\le i\le n$ then clearly M_i is $R-$module because $R^*J_i\subset J_i$. ($\because R$ Means R^*-module).We

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can consider M is an $R-$module with the induced structure. We Claim $M = M_1 \oplus M_2 \oplus M_3 \oplus M_4 \oplus \cdots \oplus M_n$ (as $R-$module). Let $M = M_1 + M_2 + \ldots + M_n$. Now $M = R_n M = J_1 M + J_2 M + J_3 M + \cdots + J_n M$ $= M_1 + M_2 + M_3 + \cdots + M_n$. **Sum is direct**; Let $x_i \in M_i$ for $1 \le i \le n$ be such that $x_1 + x_2 + x_3 + \cdots\cdots + x_n = 0$. Claim: $x_i = 0, \ \forall \ i = 1, 2, \ldots, n.$ Let $1 \le p \le n$, we prove that $x_p = 0$. Now $x_p = -(x_1 + x_2 + x_3 + \cdots + x_p + \cdots + x_n)$, If $i \ne p\{1, 2, \ldots, n\}$ then $I_i x_p \in I_i M_p = I_i J_p M = (0)$ $\Rightarrow I_i x_p = 0, \forall \ i \in \{1, 2, \ldots, n\} \ \{p\}$ Also, $I_p x_p = I_p\left[-\{x_1 + x_2 + \ldots + x_p + \ldots + x_n\}\right] = (0) \Rightarrow I_i x_p = 0, \ \forall \ i = 1, 2, \ldots, n. \Rightarrow R_n x_p = (0) \ \Rightarrow x_p = 0$.Thus $x_i = 0, \ \forall i = 1, 2, \ldots, n.$ So, $M = M_1 \oplus M_2 \oplus M_3 \oplus M_4 \oplus \cdots\cdots \oplus M_n$

$$A = ((a_{ij})) \in R_n, \ 1 \le p \ne q \le n, \quad A^{(p,q)} = \left((a_{ij}^{(pq)})\right), \quad a_{ij}^{(pq)} = \begin{cases} a_{ij} & if \quad j \ne p, \ne q \\ a_{ip} & if \quad j = q \\ a_{iq} & if \quad j = p \end{cases}$$

If $A \in J_p$ then $A^{(p,q)} \in J_q$, $\quad j \ne q, \ \left(a_{ij}^{(p,q)}\right) = \begin{cases} a_{iq} = 0, & j = p \\ a_{ij} = 0, & j \ne p \end{cases}$

$1 \le p \ne q \ne r \le n, \ A \in J_p, \ \left(A^{(p,q)}\right)^{(q,r)} = A^{(p,r)} \quad (Note: \left(A^{(p,q)}\right)^{(p,q)} = A)$. r^{th} column of $A^{(p,r)}$ is p^{th} column of A, $s \ne r$, j^{th} column of $A^{(p,r)}$ is 0 (zero) for $1 \le i, \ j \le n$ we define $R-$isomorphism $f_{ij} = M_i \to M_j$.We take $f_{ii} = 1d_{M_i}$.So, let $1 \le p \ne q \le n$ and define a map $f_{pq} : M_p \to M_q$, as follows: Let $x \in M_p = J_p M$ then $x = A_1 x_1 + A_2 x_2 + A_3 x_3 + \cdots + A_k x_k, \ x_i \in M_i, \ A_i \in J_p, 1 \le i \le n$.

Define $f_{pq}(x) = A_1^{(p,q)} x_1 + A_2^{(p,q)} x_2 + A_3^{(p,q)} x_3 + \cdots\cdots + A_k^{(p,q)} x_k \in J_q M = M_q$. f_{pq} ***is well defining map:*** Suppose, $x = A_1 x_1 + A_2 x_2 + A_3 x_3 + \cdots\cdots + A_k x_k = B_1 y_1 + B_2 y_2 + B_3 y_3 + \cdots\cdots + B_l y_l$
where $x_i, \ y_i \in M_i$, $A_i, \ B_j \in J_p$, $1 \le i \le k$, $1 \le j \le l$, implies
$A_1 x_1 + A_2 x_2 + A_3 x_3 + \cdots + A_k x_k - B_1 y_1 - B_2 y_2 - B_3 y_3 - \cdots - y_l B_l = 0$.

Claim: $A_1^{(p,q)} x_1 + A_2^{(p,q)} x_2 + A_3^{(p,q)} x_3 + \cdots\cdots + A_k^{(p,q)} x_k = B_1^{(p,q)} y_1 + B_2^{(p,q)} y_2 + B_3^{(p,q)} y_3 + \cdots\cdots + B_l^{(p,q)} y_l$
Let, $m = A_1 x_1 + A_2 x_2 + A_3 x_3 + \cdots\cdots + A_k x_k - B_1 y_1 - B_2 y_2 - B_3 y_3 - \cdots\cdots - y_l B_l = 0$. We need only prove that $m = 0$. For this it suffices to prove that $R_n m = (0)$. For this we need to prove that $\left(rE_{uv}\right) m = 0, \ \forall \ r \in R, \ 1 \le u, \ v \le n$. Let $r \in R, 1 \le u, \ v \le n.$ we first note that, if $A \in J_q$ then, $(rE_{uv})A = (0)$. If $u \ne q \ (\because rE_{uv} \in I_u \ and \ I_u J_q = (0) \ if \ u \ne q)$.So, $(rE_{uv})m = 0 \ if \ u \ne q \ (\because A^{(pq)}, \ B^{(pq)} \in J_q, \ \forall \ i = 1, \ldots, k; \ j = 1, \ldots, l)$. So, we need to prove that, $(rE_{qv})m = 0$ .But $(rE_{pv})\left(A_1 x_1 + A_2 x_2 + A_3 x_3 + \cdots + A_k x_k - B_1 y_1 - B_2 y_2 - B_3 y_3 - \cdots - y_l B_l\right) = 0$
We note that $A \in R_n$, then $A(rE_{pj}) = (rE_{qj})A^{(pq)}$.

Clearly, $\left(A_1 x_1 + A_2 x_2 + A_3 x_3 + \cdots + A_k x_k - B_1 y_1 - B_2 y_2 - B_3 y_3 - \cdots - y_l B_l\right) = 0$.

Thus $(rE_{uv})m = 0, \ \forall r \in R, 1 \le u, v \le n. \Rightarrow R_n m = 0 \Rightarrow m = 0$. Clearly, f_{pq} is $R-$homomorphism. Hence, f_{pq} is $R-$iso-morphism.. Clearly, $f_{qr} \cdot f_{pq} = f_{pr}$ $\left[\because \left(A^{(pq)}\right)^{(qr)} = A^{(pr)}\right]$.

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Clearly, $M_i \neq (0), \forall\ i = 1,\cdots,n$ $(\because\ M_i = J_i M$ at least one element present in $J_i M)$.

Claim: M_1 is simple faithful left $R-$module. Let $M_1 = RM_1$. Now $M_1 \subseteq M = R_n M = (R_n M)_n = R_n^2 M$
$= (J_1 + J_2 + \ldots + J_n)\ (I_1 + I_2 + \ldots + I_n)M$
$\subseteq J_1 I_1 M + J_2 I_2 M + \cdots + J_n I_n M$ $(\because\ J_i I_i = 0, \forall\ i \neq j \in \{1,\ldots,n\}) = RM_1 + RM_2 + \cdots + RM_n$

In particular case, if $n = 2$; $J_1 I_1 M \subseteq RM_1 + RM_2$ Let $x \in M, \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}\begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} x = \begin{pmatrix} ac & ad \\ bc & bd \end{pmatrix} x =$
$\left\{\begin{pmatrix} ac & 0 \\ bc & 0 \end{pmatrix} + \begin{pmatrix} 0 & ad \\ 0 & bd \end{pmatrix}\right\} x = \left\{\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}\begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} + \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix}\begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix}\right\} x = \left\{\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}\begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} x + \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix}\begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} x\right\}$
$\in MJ_1 R + MJ_2 R = M_1 R + M_2 R$.In general $\sum_{i=1}^{k}\begin{pmatrix} ai & 0 \\ bi & 0 \end{pmatrix}\begin{pmatrix} ci & di \\ 0 & 0 \end{pmatrix} xi$. Thus $M_1 \subseteq RM_1 + RM_2 + \cdots + RM_n$. Since, the sum on RHS is direct it follows that $M_1 = RM_1$.

Claim: $(M_1)R$ **is faithful:** Let, $a \in R$, be such that $aM_1 = (0) \Rightarrow$ $aM_i = (0)$, $\forall\ i = 1, 2, \ldots, n$, this impliesw $(aI)M = (0) \Rightarrow$ $aI = 0 \Rightarrow a = 0$. **Claim:** $(M_1)R$ **is simple.** Let N be an $R-$ sub module of M_1.

Claim: $N_1 + f_{12}(N) + f_{13}(N) + \ldots + f_{1n}(N)$ is an $R-$sub module of M. $(\because\ N_1 \subseteq M_1,\ f_{12}(N) \subseteq M_2,\ \ldots,\ f_{1n}(N) \subseteq M_n)$. In particular case $n = 2$, enough to prove that $NJ_1 + f_{12}(N)J_2 \subseteq N_1 + f_{12}(N)$, $(\because\ N \subseteq M_1 = J_1 M\ ,\ f_{12}(N) \subseteq M_2 = J_2 M)$
$\Rightarrow\ I_2 N = (0)$ *and* $I_1\ f_{12}(N) = (0)$, $x \in N,$ *and* $y \in f_{12}(N)$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in R_2$ then
$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(x + y) = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix} y$. Now $x \in N \subseteq M_1 = J_1 M, y \in f_{12}(N) \subseteq M_2 = J_2 M$.
We may assume, $x = \begin{pmatrix} p & q \\ 0 & 0 \end{pmatrix} u$ *and* $y = \begin{pmatrix} 0 & 0 \\ r & s \end{pmatrix} v$, for some $u, v \in M,\ p, q, r, s \in R$
So, now $\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} x = \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}\begin{pmatrix} p & 0 \\ q & 0 \end{pmatrix} u = \begin{pmatrix} ap & aq \\ bp & bq \end{pmatrix} u = \begin{pmatrix} pa & qa \\ 0 & 0 \end{pmatrix} u + \begin{pmatrix} 0 & 0 \\ bp & bq \end{pmatrix} u$
$= \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} p & q \\ 0 & 0 \end{pmatrix} u + \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} 0 & 0 \\ p & q \end{pmatrix} u = ax + bf_{12}(x) \in N + f_{12}(N)$. Similarly, $y\begin{pmatrix} 0 & c \\ 0 & d \end{pmatrix} \in N + f_{12}(N)$.
So, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}(x + y) \in N + f_{12}(N)$. Thus, $N + f_{12}(N) + \ldots + f_{1n}(N)$ is an $R_n -$sub module of M.
Since by hypothesis M is an $R_n -$simple it follows that either $N + f_{12}(N) + \ldots + f_{1n}(N) = (0)$ **Or** M
$\Rightarrow N = (0)$ *or* $N = M_1$ .This proves our claim that M^n is faithful $M_n(R)-$module. Thus, M^n is simple faithful $M_n(R)-$module. So, $M_n(R)$ is a left primitive ring.

Theorem: 1.12. For a ring R the following conditions are equivalent.
(i) R is left primitive
(ii)There exists a maximal left ideal $\aleph$ of R such that $(\aleph : R) = 0$.

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Proof: $(i) \Rightarrow (ii)$ By hypothesis, there exists a simple, faithful $R-$ module, say, M. By Proposition 3.9, M isomorphic to the factor module $R/\aleph$ for some modular maximal left ideal $\aleph$ of R. Then, by the remark after Definition 3.5, $0 = ann_R(M) = ann_R(R/\aleph) = (\aleph : R)$. This proves our required assertion.

(ii)$\Rightarrow$ (i):By hypothesis, there exists a maximal left ideal $\aleph$ of R such that $(\aleph : R) = 0$.Consider the factor module $R/\aleph$. As observed above, by hypothesis, $ann_R(R/\aleph) = \aleph : R = 0$. We need only prove that $R/\aleph$. By the Correspondence Theorem, $R/\aleph$, has no nontrivial sub modules. It remains to prove that $R(R/\aleph) \neq 0$. Suppose the contrary. Then $R \subseteq ann_R(R/\aleph) = (\aleph : R) = 0$, by hypothesis, which forces $R = 0$, which is impossible. So $R(R/\aleph) \neq 0$ which proves that $R/\aleph$ is simple. Thus, $R/\aleph$ is a simple, faithful $R-$ module. This proves that R is left primitive.

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