

Study of Euler's Method

Samarth Gosavi¹, Rushi Ambure¹, Vishal V. Mehtre²

Student of B.Tech, Department of Electrical Engineering, Bharati Vidyapeeth (Deemed To Be University) College of Engineering, Pune, Maharashtra, India¹

Assistant Professor, Department of Electrical Engineering, Bharati Vidyapeeth (Deemed To Be University) College of Engineering, Pune, Maharashtra, India²

Abstract: The main objective of this paper is to explain the comparison between Euler's method and the Modified Euler's Method to solve the Ordinary Differential Equation (ODE) numerically and their applications in different fields of engineering. Euler and Modified Euler techniques have been implemented using different step size and time. Euler's method is used for approximating solutions to certain differential equations and works by approximating a solution curve with line segments.

Keywords: Euler's method, Modified Euler's method.

INTRODUCTION

In mathematics and computational science, the Euler method (also called forward Euler method) is a first-order numerical procedure for solving ordinary differential equations (ODEs) with a given initial value.

The Euler method is named after Leonhard Euler, who treated it in his book *Institutionum calculi Integralis* (published 1768–1870).[1]

Euler's method is one of the simplest among the numerical methods to solve the differential equation. But this method is very slow, and the number of errors is more in the solution [2]. So, to overcome this problem many mathematicians modified the Euler's method to reduce the errors [3][4]. The two most popular methods are Mid-point method and Modified Euler's method. In this review paper we have focused on Euler's method and Modified Euler's method.

Derivation

Consider the differential equation

$$\frac{dy}{dx} = f(x, y), y(x_0) = y_0 \quad \dots(1)$$

Suppose we wish to approximate the solution to the initial-value problem (1) at $x = x_1 = x_0 + h$, where h is small. The idea behind Euler's method is to use the tangent line to the solution curve through (x_0, y_0) to obtain such an approximation. (See Figure 1.1.)

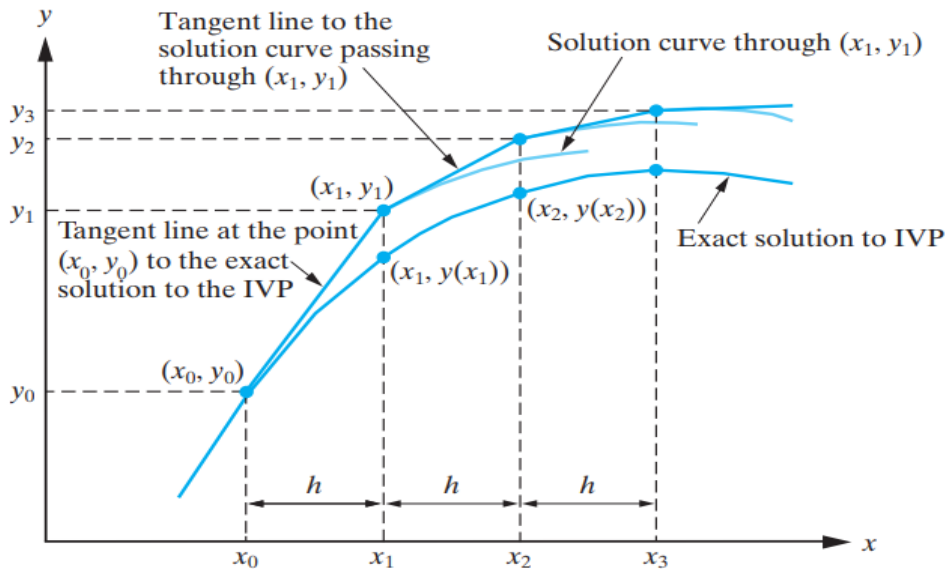
The equation of the tangent line through (x_0, y_0) is

$$y(x) = y_0 + m(x - x_0),$$

where m is the slope of the curve at (x_0, y_0) . From Equation (1), $m = f(x_0, y_0)$, so

$$y(x) = y_0 + f(x_0, y_0)(x - x_0).$$

Numerical solution of first order differential equation



Euler's method for approximating the solution to the initial-value problem $dy/dx = f(x, y), y(x_0) = y_0$.

Setting $x = x_1$ in this equation yields the Euler approximation to the exact solution at x_1 , namely,

$$y_1 = y_0 + f(x_0, y_0)(x_1 - x_0),$$

which we write as

$$y_1 = y_0 + hf(x_0, y_0).$$

Now suppose we wish to obtain an approximation to the exact solution to the initial value problem (1.1) at $x_2 = x_1 + h$. We can use the same idea, except we now use the tangent line to the solution curve through (x_1, y_1) . From (1.1), the slope of this tangent line is $f(x_1, y_1)$, so that the equation of the required tangent line is

$$y(x) = y_1 + f(x_1, y_1)(x - x_1).$$

Setting $x = x_2$ yields the approximation

$$y_2 = y_1 + hf(x_1, y_1),$$

where we have substituted for $x_2 - x_1 = h$, to the solution to the initial-value problem at $x = x_2$. Continuing in this manner, we determine the sequence of approximations

$$y_{n+1} = y_n + hf(x_n, y_n), n = 0, 1, \dots$$

to the solution to the initial-value problem (1.10.1) at the points $x_{n+1} = x_n + h$. In summary, Euler's method for approximating the solution to the initial-value problem

$$y' = f(x, y), y(x_0) = y_0$$

at the points $x_{n+1} = x_0 + nh$ ($n = 0, 1, \dots$) is

$$y_{n+1} = y_n + hf(x_n, y_n), n = 0, 1, \dots \quad (1.2)$$

Program of Euler's method :

```
%Eulers Method
clear
```

```
clc
f=input('\n Enter the function f(x,y):'); %inline('x-y/2')
% Write your f(x,y) function, where dy/dx=f(x,y), x(x0)=y0.
% f=@ (x,y) x-y/2;
x0=input('\n Enter initial value of x i.e. x0: '); %example x0=0
y0=input('\n Enter initial value of y i.e. y0: '); %example y0=1
xn=input('\n Enter the final value of x: ');
% where we need to find the value of y
%example x=0.8
h=input('\n Enter the step length h: '); %example h=0.2
%Formula: y1=y0+h*f(x0,y0);
fprintf('\n x y ');
fprintf('\n %4.3f %4.3f ',x0,y0);
while x0<xn
y1=y0+h*f(x0,y0);
x1=x0+h;
x0=x1;
y0=y1;
fprintf('\n %4.3f %4.3f ',x0,y0); %values of x and y
end
answer=y0;
fprintf('\n The value of y at x=%0.2f is y=%2.3f',xn,answer);
```

Output of program :

Enter the function f(x,y):

inline ('x-y/2')

Enter initial value of x i.e. x0: 0

Enter initial value of y i.e. y0: 1

Enter the final value of x: 0.2

Enter the step length h: 0.1

X y

0.000 1.000

0.100 0.950

0.200 0.912

The value of y at x=0.20 is y=0.912

Example:

Find $y(0.2)$ for $y' = \frac{x-y}{2}$, $y(0) = 1$, with step length 0.1 using Modified Euler method

Solution:

Given $y' = \frac{x-y}{2}$, $y(0) = 1$, $h = 0.1$, $y(0.2) = ?$

Euler method

$$y_1 = y_0 + hf(x_0, y_0)$$

$$= 1 + (0.1)f(0, 1)$$

$$= 1 + (0.1)(-0.5)$$

$$= 1 + (-0.05)$$

$$= 0.95$$

$$y_2 = y_1 + hf(x_1, y_1)$$

$$= 0.95 + (0.1)f(0.1, 0.95)$$

$$= 0.95 + (0.1)(-0.425)$$

$$= 0.95 + (-0.0425)$$

$$= 0.9075$$

$$\therefore y(0.2) = 0.9075$$

Modified Euler's method (Heun's Method)

Recall Euler's formula i.e.,

$$y_{n+1} = y_n + hf(x_n, y_n)$$

Here,

$$f(x_n, y_n) = \phi'(x_n) \text{ ie slope at } (x_n, y_n)$$

Since step size h is multiplied by slope at (x_n, y_n) , the value y_{n+1} contains large errors

From Euler's formula of equation above, we get the next value of y' i.e., y_{n+1} and

$$x_{n+1} = x_n + h$$

Thus $f(x_{n+1}, y_{n+1})$ is the slopes at (x_{n+1}, y_{n+1})

Let's take average of the slopes at (x_n, y_n) and (x_{n+1}, y_{n+1}) as,

$$\text{Average slope} = \frac{f(x_n, y_n) + f(x_{n+1}, y_{n+1})}{2}$$

Putting this slope in Euler's formula

$$y_{n+1} = y_n + h \left[\frac{f(x_n, y_n) + f(x_{n+1}, y_{n+1})}{2} \right]$$

In the above equation, we need value of y_{n+1} on RHS. This can be calculated by Euler's basic formula. Thus, we have two equations for Euler's modified method.

$$y_{n+1}^{(k)} = y_n + hf(x_n, y_n)$$

This is called predictor equation

$$y_{n+1}^{(k+1)} = y_n + \frac{h}{2} [f(x_n, y_n) + f(x_{n+1}, y_{n+1}^{(k)})]$$

This is called corrector equation.

After getting y_{n+1} by predictor equations corrector equation is applied repeatedly on

y_{n+1} till it becomes correct up to required level.

Program of modified Euler's method

```
function a= eular( df )
```

```
%df = @ (x, y) x-y/2
```

```
% for calculating value of dy/dx at some particular pt using modified euler's method
```

```
% asking intial conditions
```

```
x0 = input('Enter initial value of x : ');
```

```
y0 = input ('Enter initial value of y : ');
```

```
x1 = input( 'Enter thevalue of x at which y is to be calculated : ');
```

```
tol = input( 'Enter desired level of accuracy in the final result : ');
```

```
% calaulating the value of h
```

```
n =ceil( (x1-x0)/sqrt(tol));
```

```
h = ( x1 - x0)/n
```

```
%loop for calculating values
```

```
for k = 1 : n
```

```
    X(1,1) = x0; Y (1,1) = y0;
```

```
    X( 1, k+1) = X(1,k) + h;
```

```
    y_t = Y(1,k) + h* feval( df , X(1,k) , Y(1,k));% Euler's formula
```

```
    Y(1 ,k+1) = Y(1,k) + h/2* (feval( df , X(1,k) , Y(1,k)) + feval( df , X(1,k) + h , y_t ) );
```

```
%improving results obtained by modified Euler's formula
```

```
    while abs( Y(1,k+1) - y_t ) > h
```

```
        y_t = Y(1,k) + h*feval( df , X(1,k) , Y(1,k+1));
```

```
        Y( 1 ,k+1) = Y(1,k) + h/2* (feval( df , X(1,k) , Y(1,k)) + feval( df , X(1,k) + h , y_t ) );
```

```
    end
```

```
%displaying results
```

```
fprintf( 'for \t x = %g \n \ty = %g \n', x1,Y(1,n+1))
```

```
%displaying graph
```

```
x = 1:n+1;  
y = Y(1,n+1)*ones(1,n+1) - Y(1,:);  
plot(x,y,'b')  
xlabel = (' no of interval ');  
ylabel = (' Error ');
```

Output of program:

```
Enter the initial value of x: 0  
Enter the initial value of y: 1  
Enter the value at which y is to be calculated: 0.2  
Step length h=0.1  
Ans = 0.9146
```

Example

1. Find $y(0.2)$ for $y' = \frac{x-y}{2}$, $y(0) = 1$, with step length 0.1 using Modified Euler method

Given $y' = \frac{x-y}{2}$, $y(0) = 1$, $h = 0.1$, $y(0.2) = ?$

Here, $x_0 = 0$, $y_0 = 1$, $h = 0.1$

$$y' = \frac{x-y}{2}$$
$$\therefore f(x, y) = \frac{x-y}{2}$$

Modified Euler method

$$y_{m+1} = y_m + hf \left(x_m + \frac{1}{2}h, y_m + \frac{1}{2}hf(x_m, y_m) \right)$$

$$f(x_0, y_0) = f(0, 1) = -0.5$$

$$x_0 + \frac{1}{2}h = 0 + 0.12 = 0.05$$

$$f(x_0 + 12h, y_0 + 12hf(x_0, y_0)) = f(0.05, 0.975) = -0.4625$$
$$y_1 = y_0 + hf \left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}hf(x_0, y_0) \right)$$
$$= 1 + 0.1 \cdot -0.4625 = 0.95375$$

$$\therefore y(0.1) = 0.95375$$

Again taking (x_1, y_1) in place of (x_0, y_0) repeat the process

$$f(x_1, y_1) = f(0.1, 0.95375) = -0.42688$$

$$x_1 + \frac{1}{2}h = 0.1 + 0.12 = 0.15$$

$$y_1 + \frac{1}{2}hf(x_1, y_1) = 0.95375 + 0.12 \cdot -0.42688 = 0.93241$$

$$f(x_1 + 12h, y_1 + 12hf(x_1, y_1)) = f(0.15, 0.93241) = -0.3912$$

$$y_2 = y_1 + hf \left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}hf(x_1, y_1) \right) = 0.95375 + 0.1 \cdot -0.3912 = 0.91463$$

$$\therefore y(0.2) = 0.91463$$

Comparison of Euler’s method and Modified Euler’s method

Euler’s method	Modified Euler’s method
In the Euler's method, for calculation of $y_1 = y_0 + \Delta y$, we consider the slope at the point 0. Thus, for the calculation of y at $n+1$, we consider the slope at the point n .	In modified Euler’s method, the slope considered is the average of the slope at the points 0 and 1 where points 0 and 1 are the end points for each sub - interval.
One of the fundamental methods of numerical method is Euler’s method.[5] Euler's method approaches various ways, namely by using the Taylor series and using quadrilateral rule.	In this method, the solution we had in Euler’s method was used as an initial estimate (predictor) and then corrected using modified Euler's method.[6]
Approximation error is proportional to the step size h . [7] Hence good approximation is obtained with a very small h . [8] This requires many times discretization leading to a large computation time.	It requires less error than Euler’s method. Hence good approximation is obtained with a very small h . [9]
It is less accurate and numerically unstable.	It is accurate for numerical problems.
Output of example is 0.9075, this answer is less accurate.	Output of example is 0.9146, so the difference between Euler's and modified euler’s answer is 0.0071.

Application

- Euler's method is used for approximating solutions to certain differential equations and works by approximating a solution curve with line segments. [10] In the image to the right, the blue circle is being approximated by the red line segments.
- In some cases, it's not possible to write down an equation for a curve, but we can still find approximate coordinates for points along the curve by using simple lines [11]. These line segments have the same slope as the curve, so they stay relatively close to it.
- Euler's method is useful because differential equations appear frequently in physics, chemistry, and economics, but usually cannot be solved explicitly, requiring their solutions to be approximated. For example, Euler's method can be used to approximate the path of an object falling through a viscous fluid, the rate of a reaction over time, the flow of traffic on a busy road, to name a few.

RESULT AND DISCUSSION:

In this review paper we have used Euler’s method and Modified Euler’s method to solve Ordinary Differential Equation (ODE). While solving the Euler's method we were getting the answer easily, but the answer was not accurate and there was error in the answer. So, to get accurate answer we used Modified Euler’s method, by using this method there were less errors in the answer.

Euler's Method	Modified Euler’s Method
$y_1 = 0.95$	$y_1 = 0.9537$
$y_2 = 0.9075$	$y_2 = 0.9146$

In the above table we can clearly see the difference between the answers of Euler’s and Modified Euler’s Method. The error between both answers is 0.0071, so to get accurate answer we use Modified Euler’s Method.

CONCLUSION

The purpose of this paper was to know the difference between the Euler's method and the Modified Euler's method. Euler's method which is the most fundamental method to solve differential equation but there are much more errors in solution and using the Modified Euler's method the error in the solution was reduced.

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