

# ORDER REDUCTION OF LINEAR DYNAMIC DISCRETE SYSTEMS USING IMPROVED GENERALISED LEAST-SQUARES METHOD AND DIFFERENTIAL EVOLUTION ALGORITHM

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**Abstract:** The authors present an algorithm for order reduction of linear dynamic SISO discrete systems using the combined advantages of the improved generalised Least squares method and error minimization by Differential Evolution technique (DE). The denominator of the reduced order model is obtained by improved generalise least squares method and the numerator coefficients are determined by minimizing the integral square error between the transient responses of original and reduced order models using DE technique, pertaining to unit step input. The reduction procedure is simple, efficient and computer oriented. The algorithm is illustrated with the help of two numerical examples to highlight the advantages of the approach and the results are compared with the other existing techniques.

**Keywords:** Improved Least Squares, Integral square error, Order Reduction, Differential algorithm.

## I. INTRODUCTION

Mathematical modelling of most physical systems is carried out using theoretical considerations. This modelling procedure leads to high order state space model in time domain or state space representation and a high order transfer function model in frequency domain representation. It is often desirable to represent such models by equivalent lower order state variable or transfer function models for control and other purposes. Several methods [1]-[5] have been proposed for solving the model approximation problem and they may be grouped into two major categories, the performance-oriented and the non performance-oriented approaches. In using a model approximation method that is not performance-oriented, the original system model is first transformed into a canonical form. For a performance oriented one, approximate models are obtained by minimizing certain approximation error criteria, such as  $H^2$  - norm [10]. Differential evolution (DE) is a simple evolutionary algorithm [11] that mutates vectors by adding weighted, random differentials to them. The choice of differential evolution algorithm for numerical optimization is based on its useful features [12]-[13].

## II. DESCRIPTION OF PROBLEM

Consider an  $n^{\text{th}}$  order linear time invariant discrete system represented by

$$G_n(z) = \frac{N(z)}{D(z)} = \frac{A_0 + A_1z + \dots + A_{n-1}z^{n-1}}{B_0 + B_1z + \dots + B_{n-1}z^{n-1} + B_nz^n} \quad (1)$$

Apply linear transformation  $z = 1 + p$  to  $G_n(z)$ , then

$$G_n(p) = \frac{N(p)}{D(p)} = \frac{a_0 + a_1p + \dots + a_{n-1}p^{n-1} + a_np^n}{b_0 + b_1p + \dots + b_{n-1}p^{n-1} + b_np^n}$$

The objective is to find an  $r^{\text{th}}$  order model that has a transfer function ( $r < n$ ):

$$R(p) = \frac{N_r(p)}{D_r(p)} = \frac{c_0 + c_1p + \dots + c_{r-1}p^{r-1}}{d_0 + d_1p + \dots + d_{r-1}p^{r-1} + p^r} = \frac{\sum_{i=0}^{r-1} c_i p^i}{\sum_{j=0}^r d_j p^j}$$

Apply inverse linear transformation  $p = z - 1$  to  $R(p)$ , then

$$R(z) = \frac{N_r(z)}{D_r(z)} = \frac{C_0 + C_1z + \dots + C_{r-1}z^{r-1} + C_rz^r}{D_0 + D_1z + \dots + D_{r-1}z^{r-1} + D_rz^r} = \frac{\sum_{i=0}^r C_i z^i}{\sum_{j=0}^r D_j z^j} \quad \dots (2)$$

Where

$A_i (0 \leq i \leq n - 1)$ ;  $B_j (0 \leq j \leq n)$  &  $C_i (0 \leq i \leq r)$ ;  $D_j (0 \leq j \leq r)$  are scalar constants.

The derivation of successful reduced model  $R(s)$  for the original higher order model  $G_n(s)$  is done by minimizing the error index 'E', known as ISE, employing Generalized Least Squares method for denominator and Differential Evolution Algorithm for numerator and is given by:

$$E = \int_0^\infty [G_n(t) - R(t)]^2 dt \quad \dots (3)$$

Where  $G_n(t)$  and  $R(t)$  are the unit step response of original and reduced order systems.

### III. DIFFERENTIAL EVOLUTION

#### A. OVERVIEW

Let  $f(x)$  be a function of  $n$ -parameter vector

$$x = [x_1, x_2, \dots, x_n]^T \quad (4)$$

Determine the values of the parameter vector within the intervals

$$x_k \in (x_k^-, x_k^+), k=1,2,\dots,n \quad (5)$$

which minimize the cost function  $f(x)$ .

Differential evolution (DE) is a simple evolutionary algorithm for parameter optimization [9]. The most distinct feature of DE is that it mutates vectors by adding weighted, random vector differentials to them. Usually, the performance of a DE algorithm depends on three variables: the population size  $N_p$ , the mutation scaling factor  $F_s$ , and the crossover rate  $C_r$ . In applying a DE algorithm to solve the above parameter optimization problem, it starts by generating a population of  $N_p$  real valued  $n$ -dimensional vectors whose initial parameter values being chosen at random from within bounds set by the user. This population undergoes evolution in a form of natural selection. In every generation, each vector in the population becomes a target vector. Each target vector crosses over with a donor vector, which is generated by mutating a randomly-selected population vector with the difference between two randomly-selected population vectors, in order to produce a trial vector. If the cost of the trial vector is less than that of the target, the target is replaced by the trial vector in the next generation.

#### B. POPULATION INITIALIZATION

The DE algorithm is started with generating a population of  $N_p$  real-valued  $n$ -dimensional vectors

$$x_j = [x_{j,1}, x_{j,2}, \dots, x_{j,n}]^T, j=1,2,\dots,N_p \quad (6)$$

whose initial parameter values are chosen at random from within user-defined bounds

$$x_{j,k} \in [\underline{x}_k, \bar{x}_k], k=1,2,\dots,n \quad (7)$$

It is noted that the search region or interval  $[\underline{x}_k, \bar{x}_k]$  set by a user for the parameter  $x_k$  may be smaller than, but within, the allowable interval  $(x_k^-, x_k^+)$  in the case  $x_k^- = -\infty$  of and/or  $x_k^+ = \infty$ . Once the initial population is generated, the cost of each population vector is evaluated and stored for future reference.

#### C. MUTATION

Mutation is an operation that adds a vector differential to a population vector. In the DE algorithm, a population vector  $x_\alpha$  is mutated into  $z = [z_1, z_2, \dots, z_n]^T$  by adding to  $x_\alpha$  the weighted difference of two randomly selected but different population vectors  $x_\beta$  and  $x_\gamma$ , i.e.,

$$z = x_\alpha + F_s^*(x_\beta + x_\gamma) \quad \dots (8)$$

where  $F_s$  is a scaling factor in the interval (0,2). In the event that mutation causes a parameter  $z_k$  to shift outside the allowable interval  $(x_k^-, x_k^+)$  then  $z_k$  is set to  $x_k^-$  if

$z_k < x_k^-$ , and  $x_k^+$  if  $z_k > x_k^+$ . The mutated vector  $z$  will be used as a donor vector for producing a trial vector.

It is noted the mutation scheme (8) makes the DE a local optimizer because the vector differentials generated by a converging population eventually become infinitesimal.

#### D. CROSSOVER

Crossover is used to generate a trial vector by replacing certain parameters of the target vector by the corresponding parameters of a randomly generated donor vector. The crossover rate  $C_r$  determines when a parameter should be replaced. The process of producing a trial vector  $x_t = (x_{t,1}, \dots, x_{t,n})^T$  from the target vector  $x = (x_1, \dots, x_n)^T$  and the donor vector  $z$  is begin with generating a set of  $n$  random numbers  $\{r_1, r_2, \dots, r_n\}$  which are distributed uniformly in the interval (0,1). Next, a set of non uniform binary sequence  $b_1, b_2, \dots, b_n$  is generated by letting

$$b_i = \begin{cases} 1 & \text{if } r_i \leq C_r \\ 0 & \text{otherwise} \end{cases} \quad i=1,2,\dots,n$$

Then each element  $x_{t,k}$  of the trial vector  $x_t$  is taken as

$$x_{t,k} = \begin{cases} x_k & \text{for } b_k = 1 \\ z_k & \text{for } b_k = 0 \end{cases} \quad (9)$$

Once the trial vector has been determined, its cost is evaluated and compared with that of the corresponding target vector. The target vector will be replaced by it in the next generation if its cost is larger than that of the trial vector.

#### E. SEARCH-SPACE EXPANSION SCHEME

In the case that the initial search region is set excessively large, the convergence of DE search may become very slow and is prone to get trapped in a local optimum. On the other hand, if the initial search region is set too small, it may fail to locate the true optimum, though the mutation formula with a proper scaling factor  $F_s$  allows the DE algorithm to locate the minimum that lies outside the initial search region.

Hence, in the absence of the exact knowledge about the proper and finite interval within which the true optimum parameter locates, it is desired to allow the DE algorithm to expand the search space dynamically. Toward this end, we incorporate in the DE algorithm a search space expansion scheme. The implementation of this scheme for the cases of negative allowable lower bound  $\underline{x}_k < 0$  and positive allowable upper bound  $\bar{x}_k > 0$  is described as follows.

At the end of every  $N_c$  generations of population evolution, we examine each parameter in the vector  $x^*$  that has the lowest cost. If the current upper bound  $\bar{x}_k$  of the  $k$ -th parameter  $x_k^*$  is positive and  $\bar{x}_k \geq x_k^* > \bar{x}_k/2$ , then the upper bound  $\bar{x}_k$  get doubled. In the case where is still greater than the doubled upper bound, the upper bound  $\bar{x}_k$  is set to  $\gamma x_k^*$ , where  $\gamma > 1$  is a user-set expansion factor. The selection of the search-space expansion factor  $\gamma$  depends on the size of the initial search space. Usually, it is set to 2 for a small initial search space and 1.2 for a large initial search space.

TABLE 1: VALUES OF THE DE ALGORITHM VARIABLES USED IN EXAMPLES 1 AND 2

Parameters	Values
Population Size, $N_p$	60
Maximum number of Generations, $N_i$	400
Search – space checking period, $N_c$	10
Search – space expansion factor, $\gamma$	1.2
Crossover constant, $C_r$	0.6
Mutation scaling factor, $F_s$	1.2

In the event that any parameter search interval has been expanded,  $N_p$  population vectors are generated from the expanded portion of search space and their costs are evaluated.

A new initial population of  $N_p$  population vectors is then formed by picking up the better population vectors among the  $N_p$  evolved population vectors and the  $N_p$  newly generated population vectors. Then this new initial population evolves with the expanded search space.

F. SELECTION

The cost of each trial vector  $x_t$  is compared with that of its parent target vector  $x_i$ . If the cost of the target vector  $x_i$  is lower than that of the trial vector, the target is allowed to advance to the next generation.

Otherwise, the target vector is replaced by the trial vector in the next generation.

Differential Evolution Algorithm

- 1) Choose population size  $N_p$ , mutation scaling factor  $F_s$ , search-space checking period  $N_c$ , space expansion factor  $\gamma$ , and the maximum number of iterations  $N_i$ .
- 2) Specify the initial search intervals  $[x_k, \bar{x}_k], k=1,2,\dots,n$ .
- 3) Generate randomly  $N_p$  vectors  $x_j$ , from the specified search space  $X = \{x: x_k \in [x_k, \bar{x}_k]\}, k=1,2,\dots,n$  and evaluate their costs  $C_j, j=1,2,\dots,N_p$ .
- 4) Initialize the iteration index  $I=1$ .
- 5) Set the initial target index  $j=1$ .
- 6) Choose at random three different integers  $\alpha, \beta$  and  $\gamma$  from the set  $\{j=1,2,\dots,N_p\}$ .
- 7) Mutate the vector  $x_\alpha$  to  $z = x_\alpha + F_s * (x_\beta - x_\gamma)$ .
- 8) Generate a non uniform binary Z-sequence  $b = \{b_1, b_2, \dots, b_n\}$ .
- 9) Obtain the trial vector  $x_t$  from the mutated vector Z and the target vector  $x_j$  using crossover scheme (9).
- 10) Evaluate the cost of the trial vector  $x_t$  and denote it by  $C_t$ . If  $C_t < C_j$  then  $x_j^{new} = x_t$  and  $c_j^{new} = c_t$ , otherwise  $x_j^{new} = x_j$  and  $c_j^{new} = c_j$ .
- 11) Increment the target index  $j$  by 1. If  $j < N_p$  then go to Step 6.
- 12) Do the replacement  $x_j = x_j^{new}$  and  $c_j = c_j^{new}$  for  $j=1,2,\dots,N_p$ .
- 13) If  $i$  is a multiple of  $N_c$ , then execute the search-space expansion scheme.
- 14) Increment the iteration index  $I$  by 1. If  $I < N_i$ , then go to Step 5.
- 15) Stop.

III. ORDER REDUCTION BY GENERALISED LEAST-SQUARES METHOD

Here, the model reduction by generalized least-squares method suggested in [11] is discussed in brief

Consider the  $n^{th}$  order system transfer function, given by :

$$G_n(s) = \frac{b_0 + b_1s + \dots + b_{n-1}s^{n-1}}{a_0 + a_1s + \dots + a_{n-1}s^{n-1} + a_n s^n} \tag{10}$$

For a reduced  $r^{th}$  order model of  $G_n(s)$  in (1), given by :

$$G_r(s) = \frac{d_0 + d_1s + \dots + d_{r-1}s^{r-1}}{e_0 + e_1s + \dots + e_{r-1}s^{r-1} + s^r} \tag{11}$$

which retains  $(r + t)$  time moments and  $(r - t)$  Markov parameters  $(0 \leq t \leq r)$  the coefficients  $e_k, d_k$  in (2) are derived from following set of equations :

$$\left. \begin{aligned} d_0 &= e_0 c_0 \\ d_1 &= e_1 c_0 + e_0 c_1 \\ &\vdots \\ d_{r-1} &= e_{r-1} c_0 + \dots + e_0 c_{r-1} \\ 0 &= e_{r-1} c_1 + \dots + e_1 c_{r-1} + e_0 c_r \\ 0 &= e_{r-1} c_2 + \dots + e_1 c_r + e_0 c_{r+1} \\ &\vdots \\ 0 &= e_{r-1} c_t + \dots + e_1 c_{r+t-2} + e_0 c_{r+t-1} \end{aligned} \right\} \tag{12}$$

and

$$\left. \begin{aligned} d_{r-1} &= m_1 \\ d_{r-2} &= m_1 e_{r-1} + m_2 \\ &\vdots \\ d_t &= m_1 e_{t+1} + m_2 e_{t+2} + \dots + m_{r-t} \end{aligned} \right\} \tag{13}$$

where, the  $c_i$  and  $m_j$  are the time moment proportionals and Markov parameters of the system, respectively. Elimination of the  $d_j (j = t, t + 1, \dots, r - 1)$  in (13) by substituting into (12) gives the reduced denominator coefficients as the solution of :

$$\begin{pmatrix} c_{r+t-1} & c_{r+t-2} & \dots & \dots & \dots & c_t \\ c_{r+t-2} & c_{r+t-3} & \dots & \dots & c_t & c_{t-1} \\ \vdots & \vdots & \dots & \dots & \vdots & \vdots \\ c_{r-1} & c_{r-2} & \dots & \dots & c_1 & c_0 \\ c_{r-2} & c_{r-3} & \dots & \dots & c_0 & -m_1 \\ \vdots & \vdots & \dots & \dots & \vdots & \vdots \\ c_t & c_{t-1} & \dots & c_0 & -m_1 & \dots & -m_{t-1} \end{pmatrix} \begin{pmatrix} e_0 \\ e_1 \\ \vdots \\ \vdots \\ \vdots \\ e_{r-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ m_{r-t} \end{pmatrix} \tag{14}$$

Or,  $H e = m$  in matrix vector form.

If the denominator given by  $e$  in (14) is unstable, or has a singularity, then the next Markov parameter  $m_{r-t+1}$  can be assumed to be matched by extending (13) with the equation :

$$d_{t-1} = m_1 e_{t+2} + m_2 e_{t+1} + \dots + m_{r-t+1} \tag{15}$$

This in effect adds another row to the H matrix and the m vector in (14), given by :

$[c_{t-1} \ c_{t-2} \ \dots \ c_0 \ -m_1 \ -m_2 \ \dots \ -m_{r-t}]$  and  $[m_{r-t+1}]$ , respectively. Calculation of  $e$  from this non square system of equations can only be done in the least- squares sense, i.e. :

$$e = (H^T H)^{-1} H^T m \tag{16}$$

If the denominator polynomial is still not adequate, then the H matrix and the m vector may again be extended by

assuming a matching of the next Markov parameter in the sequence and (16) is solved for the new estimate of e.

#### IV. ILLUSTRATIVE EXAMPLES

Example 1: Consider an Eighth order system transfer function

$$G(z) = \frac{0.4209z^7 + 0.2793z^6 - 0.0526z^5 + 0.038z^4 - 0.1291z^3 - 0.0656z^2 + 0.011z - 0.0015}{z^8 - 0.4209z^7 - 0.2793z^6 + 0.0526z^5 - 0.038z^4 + 0.1291z^3 + 0.0656z^2 - 0.011z + 0.0015}$$

Apply linear transformation,  $z = p+1$

$$G(p) = \frac{0.4209p^7 + 3.2256p^6 + 10.4621p^5 + 18.695999p^4 + 19.8144p^3 + 12.277499p^2 + 4.0036p + 0.5004}{p^8 + 7.5791p^7 + 24.774399p^6 + 45.537903p^5 + 51.304005p^4 + 36.185604p^3 + 15.722501p^2 + 3.9964p + 0.4996}$$

Denominator by generalized least square method:

No. of time moments=3

$t[0]=1.001601, t[1]=0.001603, t[2]=-6.958732$

No. of markov parameters=1

$m[1]=0.4209$

The second order reduced denominator using generalized least square method in p-domain is

$$D(p) = p^2 + 0.419997p + 0.144031$$

Numerator by DE technique:

No. of iterations=100

Swarm size=50

$p_l=0.1, p_u=2$

The second order reduced numerator using differential evolution technique in p domain is

$$N(p) = 0.387715p + 0.126863$$

The proposed second order reduced model obtained is

$$R(p) = \frac{0.387715p + 0.126863}{p^2 + 0.419997p + 0.144031} \quad (\text{ISE}=0.013664)$$

Apply inverse linear transformation,  $p = z-1$

$$R(z) = \frac{0.387715z - 0.260852}{z^2 - 1.580003z + 0.724034}$$

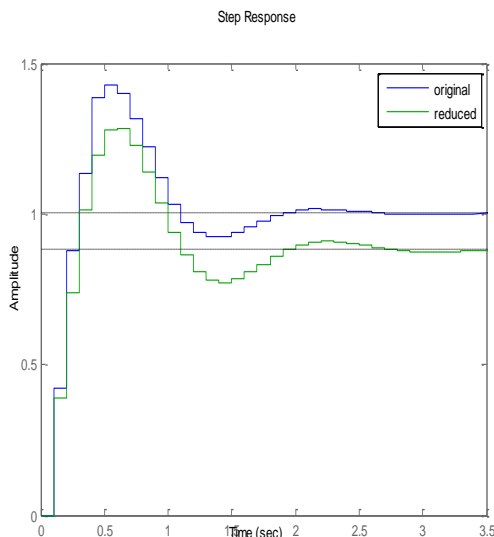


Fig. 1(a) Comparison of step responses of  $G(z)$  and  $R(z)$

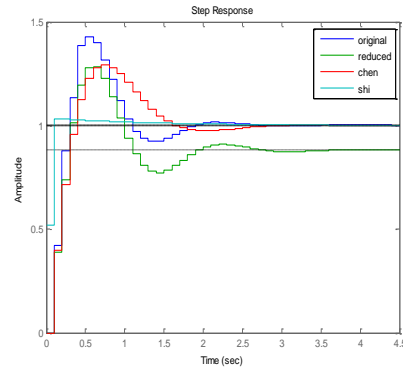


Fig. 1(b) Comparison of step responses of  $G(z)$  and  $R(z)$  with other reduction techniques

Example 2: Consider an fifth order system transfer function

$$G(z) = \frac{3.75z^5 - 13.0875z^4 + 17.80313z^3 - 11.7156z^2 + 3.688103z - 0.4358109}{z^6 - 4.25z^5 + 7.3625z^4 - 6.626875z^3 + 3.250838z^2 - 0.8181562z + 0.0817593}$$

Apply linear transformation,  $z = p+1$

$$G(p) = \frac{3.75p^5 + 5.6625p^4 + 2.953131p^3 + 0.668732p^2 + 0.06617p + 0.002262}{p^6 + 1.75p^5 + 1.1125p^4 + 0.323126p^3 + 0.045216p^2 + 0.002896p + 0.000066}$$

Denominator by generalized least square method: No. of Time moments = 4

$t[0]=34.272728, t[1]=-501.269989, t[2]=8647.485352, t[3]=-159075.3125$

No. of Markov parameters = 0

The second order reduced denominator using generalized least square method in p-domain is

$$D(p) = p^2 + 0.225215p + 0.009092$$

Numerator by DE technique:

No. of iterations = 120

Swarm size = 100

$p_l = 0.009092, p_u = 3.017862$

The second order reduced numerator using differential evolution technique in p-domain is

$$N(p) = 3.017862p + 0.306552$$

The proposed second order reduced model obtained is

$$R(p) = \frac{3.017862p + 0.306552}{p^2 + 0.225215p + 0.009092} \quad (\text{ISE}=0.608587)$$

Apply inverse linear transformation,  $p = z-1$

$$R(z) = \frac{3.017862z - 2.71131}{z^2 - 1.774785z + 0.783877}$$

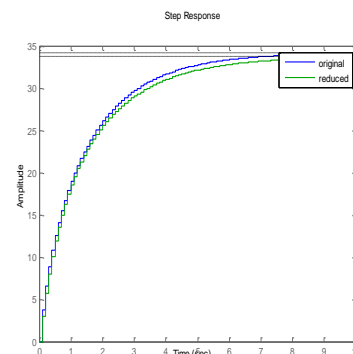


Fig. 2(a) Comparison of step responses of  $G(z)$  and  $R(z)$

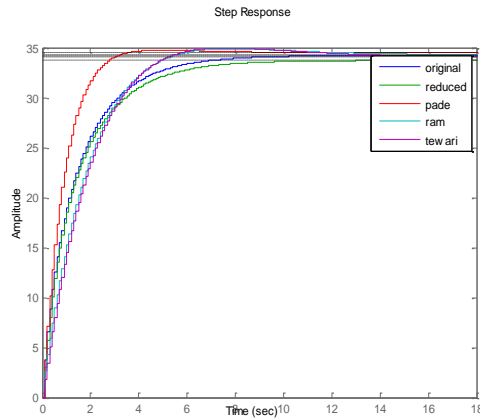


Fig. 2(b) Comparison of step responses of  $G(z)$  and  $R(z)$  with their reduction techniques

#### IV. CONCLUSION

The authors proposed a mixed algorithm for reducing the order of linear dynamic SISO systems. In this algorithm, the concept of order reduction by generalized least squares method has been improved and employed to determine the coefficients of reduced denominator while the coefficients of reduced numerator are obtained by minimizing the integral square error between the transient response of original and reduced models using DE technique pertaining to unit step input. The algorithm is implemented in C-language. The matching of the unit step response is assured reasonably well in the algorithm. The algorithm is simple and computer oriented. A comparison of step responses for the proposed reduction algorithm and the other well known existing order reduction technique is shown from which it is clear that the proposed algorithm compares well with the other techniques of model order reduction.

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